

UNIT - 2

(1)

a) Fourier SeriesFourier representation:

The signal is represented as a weighted superposition of complex sinusoids.

LTI = Linear time invariant = weighted superposition

∴ The output of LTI system can also be considered as weighted superposition of responses to the individual complex sinusoids.

orthogonality of complex sinusoidal signals:

The two signals are said to be orthogonal, if their inner product is zero.

Let $\phi_k(n)$ & $\phi_m(n)$ are two periodic signals with period N.

Then their inner product is defined as

$$I_{k,m} = \sum_{n=CN} \phi_k(n) \phi_m^*(n) \quad (\text{for discrete})$$

$$I_{k,m} = \int_{CD} \phi_k(t) \phi_m^*(t) \quad (\text{for continuous})$$

If $I_{k,m} = 0$ for $k \neq m$

Then the two signals are said to be orthogonal.

Type of Signal	<u>Fourier representation</u>
continuous time periodic signal	(CTFS) Fourier series.
Continuous time non-periodic signal	Fourier transform (CTFT)
Discrete time periodic signal	Fourier series (DTFS)
Discrete time non-periodic signal	Fourier transform (DTFT).

Applications of Fourier Series:

1. They are used analyze periodic signals.
2. they are used to analyze harmonic content of the signals.

Types of Fourier series:

1. Trigonometric Fourier series
2. compact trigonometric Fourier series / polar Fourier series
3. Exponential Fourier series.

Trigonometric Fourier series (Quadratus Fourier series)

$$x(t) = a(0) + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + \sum_{k=1}^{\infty} b_k \sin k\omega_0 t \rightarrow ①$$

Where $a(0) = \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos k\omega_0 t dt$$

$$b(k) = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin(k\omega_0 t) dt$$

Here $\int_{-T/2}^{T/2}$ indicates integration over one time period

& $\omega_0 = \frac{2\pi}{T}$ where T is the period of $x(t)$.

Compact trigonometric Fourier Series (polar form)

$$x(t) = D(0) + \sum_{k=1}^{\infty} D(k) \cos(k\omega_0 t + \phi(k)) \rightarrow ②$$

$$\text{where } D(0) = a(0) = \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt$$

$$D(k) = \sqrt{a(k)^2 + b(k)^2}$$

$$\phi(k) = \tan^{-1} \left(\frac{b(k)}{a(k)} \right)$$

Exponential Fourier Series:

$$x(t) = \sum_{k=-\infty}^{\infty} x(k) e^{jk\omega_0 t} \rightarrow \text{Synthesis Eqn.}$$

$$\text{where } x(k) = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt \rightarrow \text{Analysis Eqn.}$$

$x(k) \rightarrow$ Fourier series coefficients.

Dirichlet conditions:

1. Single valued property:

$x(t)$ must have only one value at any time instant over a finite time interval T .

2) Finite discontinuities:

$x(t)$ should have at the most finite number of discontinuities over a finite time interval T .

3) Finite peaks:

The signal $x(t)$ should have finite number of maxima & minima over a finite time interval T .

4) Absolute Integrability:

$x(t)$ must be absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

Properties of Fourier Series:

1) Linearity:

Statement: If $x(t) \xrightarrow{F.S} x(k)$ & $y(t) \xrightarrow{F.S} y(k)$

then, $z(t) = a x(t) + b y(t) \xrightarrow{F.S} z(k) = \underbrace{a x(k) + b y(k)}_{\text{super position}}$

Proof:

kkT, Analysis eqn of Exponential Fourier Series.

$$x(k) = \frac{1}{T} \int_{-T}^{T} x(t) e^{-jk\omega_0 t} dt \text{ considering this eqn}$$

We can write

$$z(k) = \frac{1}{T} \int_{-T}^{T} z(t) e^{-jk\omega_0 t} dt$$

$$E(k) = \frac{1}{T} \int_{T_0} T [a x(t) + b y(t)] e^{-jk\omega_0 t} dt \quad [z = a x(t) + b y(t)]$$

$$= \frac{1}{T} \int_{T_0} T a x(t) e^{-jk\omega_0 t} dt + \frac{1}{T} \int_{T_0} T b y(t) e^{-jk\omega_0 t} dt$$

$$z(k) = a \frac{1}{T} \int_{T_0} T x(t) e^{-jk\omega_0 t} dt + b \frac{1}{T} \int_{T_0} T y(t) e^{-jk\omega_0 t} dt$$

$$\boxed{z(k) = a x(k) + b y(k)} \quad (\because x(k) = x(t) e^{-jk\omega_0 t})$$

Hence proved.

Time shift (or) translation:

Statement: If $x(t) \xleftrightarrow{\text{Fs}} x(k)$

$$\text{then } z(t) = x(t-t_0) \xleftrightarrow{\text{Fs}} z(k) = e^{-jk\omega_0 t_0} x(k)$$

$$x(k) = \frac{1}{T} \int_{T_0} T x(t) e^{-jk\omega_0 t} dt$$

$$z(k) = \frac{1}{T} \int_{T_0} T z(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{T_0} T x(t-t_0) e^{-jk\omega_0 t} dt$$

$$\text{let } (t-t_0) = m \Rightarrow t = m + t_0$$

$$z(k) = \frac{1}{T} \int_{T_0} T x(m) e^{-jk\omega_0 (m+t_0)} dm$$

$$= \frac{1}{T} \int_{T_0} T x(m) e^{-jk\omega_0 m} e^{-jk\omega_0 t_0} dm$$

$$= e^{-jk\omega_0 t_0} \frac{1}{T} \int_{T_0} T x(m) e^{-jk\omega_0 m} dm$$

$$\boxed{z(k) = e^{-jk\omega_0 t_0} x(k)}$$

Hence proved.

Frequency shift:

Statement: If $x(t) \xleftrightarrow{F_1} x(k)$ then,

$$z(t) = e^{jk_0\omega_0 t} x(t) \xleftrightarrow{F_1} z(k) = x(k - k_0).$$

$$x(k) = \frac{1}{T} \int_{[T]} x(t) e^{-jk\omega_0 t} dt$$

$$z(k) = \frac{1}{T} \int_{[T]} z(t) e^{-jk\omega_0 t} dt$$

$$z(k) = \frac{1}{T} \int_{[T]} e^{jk_0\omega_0 t} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_{[T]} x(t) e^{-j(k-k_0)\omega_0 t} dt$$

$$\boxed{z(k) = x(k - k_0)}$$

Scaling:

If $x(t) \xleftrightarrow{F_1} x(k)$.

$$\text{then } z(t) = x(at) \xleftrightarrow{F_1} z(k) = x(k)$$

Analysis eqn of f_1 is given by

$$x(k) = \frac{1}{T} \int_{[T]} x(t) e^{jk\omega_0 t} dt$$

Since $x(t)$ is periodic, then $z(t) = x(at)$ is also

periodic.

If T' is the time period of $z(t)$ then time period

of $z(t)$ is T/a .

\therefore Fourier coefficient of $z(t)$ can be written as

$$z(k) = \frac{1}{T'} \int_{[T']} z(t) e^{-jk\omega_0 t} dt$$

$$\therefore x(k) = \frac{1}{T/a} \int_{T/a}^{\infty} x(at) e^{-jk\omega_0 at} dt$$

$$= \frac{a}{T} \int_{T/a}^{\infty} x(at) e^{-jk\omega_0 at} dt$$

put at $\equiv m$; $a dt \equiv dm \Rightarrow dt = 1/a dm$

$$x(k) = \frac{a}{T} \int_{T/a}^{\infty} x(m) e^{-jk\omega_0 m} \left[\frac{1}{a} dm \right]$$

$$x(k) = \frac{1}{T} \int_{T/a}^{\infty} x(m) e^{-jk\omega_0 m} dm$$

$$\boxed{x(k) = X(k)}$$

Time differentiation:

Statement: If $x(t) \xrightarrow{\mathcal{F}_t} X(k)$,

then,

$$\frac{d}{dt} x(t) \xrightarrow{\mathcal{F}_t} jk\omega_0 X(k) \rightarrow ①$$

proof:

By the definition of exponential Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) e^{jk\omega_0 t} \rightarrow ②$$

Differentiation Eq? ② w.r.t 't' we get,

$$\frac{d}{dt} x(t) = \sum_{k=-\infty}^{\infty} [jk\omega_0 X(k)] e^{jk\omega_0 t} \rightarrow ③$$

$$\frac{d}{dt} x(t) = jk\omega_0 X(k)$$

$$\boxed{\therefore \frac{d}{dt} x(t) \xrightarrow{\mathcal{F}_t} jk\omega_0 X(k)}$$

convolution fn time:

Statement: If $x(t) \xrightarrow{f_1} X(k)$ & $y(t) \xrightarrow{f_2} Y(k)$

then $z(t) = x(t) * y(t) \xrightarrow{f_1} Z(k) = T \cdot X(k) Y(k)$.

$$W \cdot k \cdot T \quad X(k) = \frac{1}{T} \int_{T \geq t} e^{-jk\omega_0 t} dt$$

$$Z(k) = \frac{1}{T} \int_{T \geq t} z(t) e^{-jk\omega_0 t} dt.$$

$$= \frac{1}{T} \int_{T \geq t} [x(t) * y(t)] e^{-jk\omega_0 t} dt \rightarrow 0$$

$$z(t) * y(t) = \int_{T \geq t} x(t) y(t-t) dt \rightarrow ①$$

convolution integral eqn

this convolution is perform over one period for
periodic signals. (convolution integral eqn)

sub ② in ①

$$Z(k) = \frac{1}{T} \int_{T \geq t} \left[\int_{T \geq t} x(t) y(t-t) dt \right] e^{-jk\omega_0 t} dt$$

Interchanging the order of integration

$$Z(k) = \frac{1}{T} \int_{T \geq t} x(t) \int_{T \geq t} y(t-t) e^{-jk\omega_0 t} dt dt$$

$$\text{put } (t-t) = m \Rightarrow t = t+m, dt = dm$$

Since, the integration over one period, This
Substitution will just shift the integrating limit.

Hence, we can write

$$Z(k) = \frac{1}{T} \int_{T \geq t} x(t) \int_{T \geq t} y(m) e^{-jk\omega_0 m} e^{-jk\omega_0 t} dt dm$$

$$z(k) = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt + \int_{-T/2}^{T/2} y(m) e^{-jk\omega_0 m} dm$$

$$z(k) = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt \cdot \left[\frac{1}{T} \int_{-T/2}^{T/2} y(m) e^{-jk\omega_0 m} dm \right]$$

$$z(k) = \frac{1}{T} [T x(k)] [T y(k)]$$

$$\boxed{z(k) = T x(k) y(k)}$$

Multiplication (or) modulation Theorem:

If $x(t) \xrightarrow{f_s} x(k)$ & $y(t) \xrightarrow{f_s} y(k)$.

then, $z(t) = x(t) \cdot y(t) \xrightarrow{f_s} z(k) = x(k) * y(k)$

$$x(k) = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

$$z(k) = \frac{1}{T} \int_{-T/2}^{T/2} z(t) e^{-jk\omega_0 t} dt$$

$$z(k) = \frac{1}{T} \int_{-T/2}^{T/2} x(t) y(t) e^{-jk\omega_0 t} dt$$

Synthesis Eqn of f_s is

$$x(t) = \sum_{k=-\infty}^{\infty} x(k) e^{jk\omega_0 t} \rightarrow ②$$

$$z(k) = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{m=-\infty}^{\infty} x(m) e^{jm\omega_0 t} y(t) e^{-jk\omega_0 t} dt$$

$$z(k) = \sum_{m=-\infty}^{\infty} x(m) \frac{1}{T} \int_{-T/2}^{T/2} y(t) e^{jm\omega_0 t} e^{-jk\omega_0 t} dt$$

$$z(k) = \sum_{m=-\infty}^{\infty} x(m) \frac{1}{T} \int_{-T/2}^{T/2} y(t) e^{-j(k-m)\omega_0 t} dt$$

$$= \sum_{m=-\infty}^{\infty} x(m) y(k-m)$$

WKT, convolution sum Eqn is given by.

$$\underline{z(k) = x(k) * y(k)}$$

Parseval's Theorem:

If $x(t)$ is a periodic signal with Fourier coefficient $x(k)$ then average power of signal is given by power,

$$P = \sum_{k=-\infty}^{\infty} |x(k)|^2$$

The power of the signal is given by, P -

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt$$

for periodic signal, $P = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x(t) x^*(t) dt \rightarrow \textcircled{1}$$

Synthesis eqn of exponential Fourier series is given by,

$$x(t) = \sum_{k=-\infty}^{\infty} x(k) e^{jk\omega_0 t}$$

Taking conjugate on both sides we get,

$$x^*(t) = \left[\sum_{k=-\infty}^{\infty} x(k) e^{jk\omega_0 t} \right]^*$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} x^*(k) e^{-jk\omega_0 t} \rightarrow \textcircled{2}$$

Now substitute eqn \textcircled{2} in eqn \textcircled{1} we get,

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \sum_{k=-\infty}^{\infty} x^*(k) e^{-jk\omega_0 t} dt.$$

Here $\int_{-T/2}^{T/2} = \int_{-T}^T$ ie integral of one period of $x(t)$

and interchanging the order of summation & integration

$$P = \sum_{k=-\infty}^{\infty} x^*(k) \frac{1}{T} \int_{-T}^T x(t) e^{-jk\omega_0 t} dt$$

$$P = \sum_{k=-\infty}^{\infty} x^*(k) x(k)$$

$$\boxed{P = \sum_{k=-\infty}^{\infty} |x(k)|^2}$$

Note: power of signal can be obtain by squaring & adding the magnitude of fourier coefficients.

Symmetry properties:

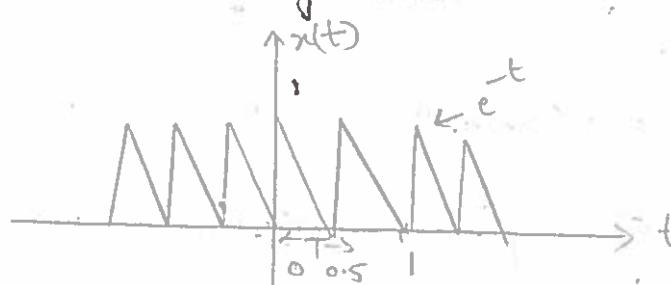
If $x(t)$ is real then, conjugate of $x^*(k) =$

If $x(t)$ is imaginary then, $x^*(k) = -x(-k)$

If $x(t)$ is real & even, imaginary $\{x(k)\} = 0$

If $x(t)$ is imaginary & odd then, real part of $\{x(k)\} = 0$

Find the trigonometric ^{fourier} series for the periodic signal shown below — figure.



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Given $T=0.5$, $x(t) = e^{-t}$

$$x(t) = a(0) + \sum_{k=1}^{\infty} a(k) \cos k\omega_0 t + \sum_{k=1}^{\infty} b(k) \sin k\omega_0 t$$

$$a(0) = \frac{1}{T} \int_{0}^{T} x(t) dt$$

$$= \frac{1}{0.5} \int_0^{0.5} e^{-t} dt$$

$$= \frac{1}{0.5} [e^{-t}]_0^{0.5}$$

$$= \frac{-1}{0.5} [e^{-0.5} - e^{-0}]$$

$$= \frac{-1}{0.5} [0.606 - 1]$$

$$= -2 [0.606 - 1]$$

$$= -2 [-0.394]$$

$$\boxed{a(0) = 0.7869}$$

$$a(k) = \frac{2}{T} \int_{0}^{T} x(t) \cos k\omega_0 t dt$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{0.5} = 4\pi$$

$$a(k) = \frac{2}{0.5} \int_0^{0.5} e^{-t} \cos 4\pi k t dt$$

$$a(k) = 4 \int_0^{0.5} e^{-t} \cos 4\pi k t dt \rightarrow ①$$

From trigonometric function $\omega k T$

$$\int e^{at} \cos bt dt = \frac{e^{at}}{a^2 + b^2} [a \cos bt + b \sin bt], a=-1, b=4\pi k \rightarrow ②$$

compare ① & ② eqns

$$a(k) = \frac{4e^{-t}}{1+(4\pi k)^2} \left[e^{0.5} \cos(4\pi k)t + 4\pi k \sin(4\pi k)t \right]_0^{0.5}$$

$$a(k) = \frac{4}{1+(4\pi k)^2} \left\{ e^{-0.5} \left[-\cos(4\pi k)0.5 + 4\pi k \sin(4\pi k)0.5 \right] \right\}$$

$$= e^{-0} \left[-\cos(4\pi k)0 + 4\pi k \sin(4\pi k)0 \right]$$

$$= \frac{4}{1+(4\pi k)^2} 0.606 \left[-\cos(2\pi k) + 4\pi k \sin(2\pi k) \right] + 1 - 0$$

$$= \frac{4}{1+(4\pi k)^2} \left[0.606 (-1+0) + (1-0) \right]$$

$$= \frac{4}{1+(4\pi k)^2} [0.606 + 1]$$

$$= \frac{4}{1+(4\pi k)^2} [0.394]$$

$$a(k) = \frac{1.576}{1+(4\pi k)^2}$$

$$b(k) = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin k \omega_0 t dt \quad (\text{here } \omega_0 = 4\pi)$$

$$= \frac{2}{0.5} \int_0^{0.5} e^{-t} \sin 4\pi k t dt$$

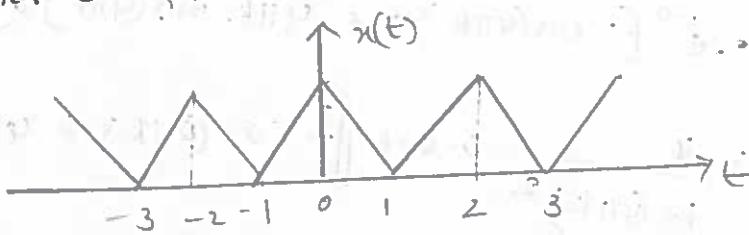
$$\int e^{an} \sin bn dt = \frac{e^{an}}{a^2+b^2} (a \sin bn + b \cos bn)$$

$$b(k) = \frac{-4e^{-t}}{1+(4\pi k)^2} \left[\sin 4\pi k t + 4\pi k \cos 4\pi k t \right]_0^{0.5} \quad (\because a=1)$$

$$\frac{4}{(1+4\pi k)^2} \cdot [e^{-0.5} (-\sin \pi k + 4\pi k \cos \pi k) - e^{-0} (-\sin 0 - 4\pi k \cos 0)]$$

$$= \frac{4}{(1+4\pi k)^2} [0.606 - 4\pi k + 4\pi k]$$

Determine the Fourier series of the function shown in the figure.



$$x(t) = \begin{cases} 2t & -1 < t < 0 \\ 0 & 0 < t < 1 \end{cases}$$

$$T=2, \omega_0 = \frac{2\pi}{T} = \frac{\pi}{2} = \pi$$

$$x(t) = \begin{cases} (t+1) & -1 < t < 0 \\ (1-t) & 0 < t < 1 \end{cases}$$

$$x(k) = \frac{1}{T} \int_{-T}^T x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{2} \left[\int_{-1}^0 (t+1) e^{-jk\omega_0 t} dt + \int_0^1 (1-t) e^{-jk\omega_0 t} dt \right]$$

$$x(k) = \frac{1}{2} \left[\int_{-1}^0 e^{-jk\omega_0 t} dt + \int_{-1}^0 t e^{-jk\omega_0 t} dt + \int_0^1 e^{-jk\omega_0 t} dt \right]$$

$$- \int_0^1 t e^{-jk\omega_0 t} dt$$

W.K.T,

$$\int_n e^{an} = e^{an} \cdot \left[\frac{1}{a} - \frac{1}{a^2} \right] ; n=t, a = -jk\omega_0$$

$$x(k) = \frac{1}{2} \left[\frac{i}{jk\omega_0} (e^{-jk\omega_0 t}) \Big|_0^1 - \frac{1}{jk\omega_0} (e^{-jk\omega_0 t}) \Big|_0^1 \right]$$

$$\left\{ e^{-jk\omega_0 t} \left(\frac{t}{-jk\omega_0} - \frac{1}{(-jk\omega_0)^2} \right) \right\}_{-1}^0 = \left\{ e^{-jk\omega_0 t} \left(\frac{t}{jk\omega_0} - \frac{1}{(jk\omega_0)^2} \right) \right\}_0^1$$

$$x(k) = \frac{1}{2} \left[\frac{-1}{jk\omega_0} (1 - e^{jk\omega_0}) - \frac{1}{jk\omega_0} (e^{-jk\omega_0} - 1) \right] +$$

$$\left[\frac{-1}{(-jk\omega_0)^2} - \left[e^{jk\omega_0} \left(\frac{1}{jk\omega_0} - \frac{1}{(-jk\omega_0)^2} \right) \right] \right]$$

$$- \left[e^{-jk\omega_0} \left(\frac{-1}{jk\omega_0} - \frac{1}{(-jk\omega_0)^2} \right) + \frac{1}{(-jk\omega_0)^2} \right]$$

$$x(k) = \frac{1}{2} \left[\left(\frac{-1}{jk\pi} + e^{\frac{jk\pi}{2}} \right) - \frac{e^{-jk\pi}}{jk\pi} + \frac{1}{jk\pi} - \frac{1}{j^2 k^2 \pi^2} \right.$$

$$\left. - \frac{e^{jk\pi}}{jk\pi} + \frac{e^{jk\pi}}{j^2 k^2 \pi^2} + \frac{e^{-jk\pi}}{jk\pi} + \frac{e^{-jk\pi}}{j^2 k^2 \pi^2} - \frac{e^{-jk\pi}}{j^2 k^2 \pi^2} \right]$$

$$x(k) = \frac{1}{2} \left[\frac{-2}{j^2 k^2 \pi^2} + \frac{e^{\frac{jk\pi}{2}}}{j^2 k^2 \pi^2} + \frac{e^{-jk\pi}}{j^2 k^2 \pi^2} \right]$$

$$= \frac{1}{2j^2 k^2 \pi^2} [-2 + e^{jk\pi} + e^{-jk\pi}]$$

$$j^2 = -1, \quad x(k) = \frac{1}{2} \cdot \frac{1}{k^2 \pi^2} (-2 + e^{jk\pi} + e^{-jk\pi})$$

$$x(k) = \frac{-1}{2} \frac{1}{k^2 \pi^2} (-2 + 2 \cos(k\pi))$$

$$x(k) = \begin{cases} \frac{2}{\pi^2}; & k = \pm 1, \pm 3, \pm 5 \\ 0; & k = \pm 2, \pm 4, \pm 6 \\ \infty; & k = 0 \end{cases}$$

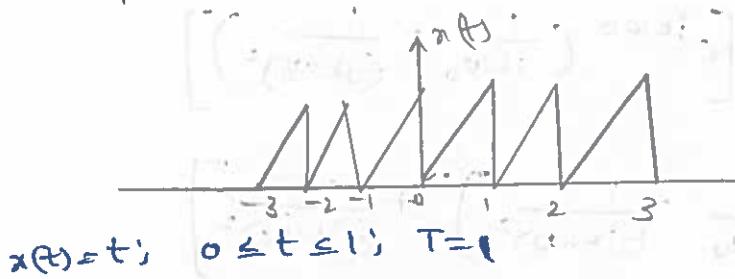
$$\begin{aligned} \cos \theta &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\ 2 \cos \theta &= e^{j\theta} + e^{-j\theta} \end{aligned}$$

$$\text{for } k=1; = \frac{-1}{2\pi^2} [-2 + 2(-1)]$$

$$\text{for } k=2; x(k) = \frac{1}{2} \left[\frac{1}{4\pi k^2} (-2+2(1)) \right]$$

$$= 0.$$

Find the exponential series Fourier series and plot the magnitude and phase spectrum for the sawtooth wave form.



$$x(t) = \sum_{k=-\infty}^{\infty} x(k) e^{-jkw_0 t}$$

Analysis eqn of exponential Fourier series is,

$$x(k) = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jkw_0 t} dt$$

$$= \frac{1}{T} \int_0^1 t e^{-jkw_0 t} dt$$

wkt,

$$\int e^{ax} dx = e^{ax} \left[\frac{1}{a} - \frac{1}{a^2} \right]; a = -jk\omega_0 \text{ & } x=t$$

$$x(k) = \left\{ e^{-jkw_0 t} \left[\frac{t}{-jk\omega_0} - \frac{1}{(-jk\omega_0)^2} \right] \right\}_0^1$$

$$= \left[e^{-jkw_0} \left[\frac{1}{-jk\omega_0} - \frac{1}{(-jk\omega_0)^2} \right] + \left[e^{jkw_0} \left[\frac{0}{-jk\omega_0} - \frac{1}{(-jk\omega_0)^2} \right] \right] \right]$$

$$= \left[e^{-jkw_0} \left(\frac{-1}{jk\omega_0} - \frac{1}{(-jk\omega_0)^2} \right) + \frac{1}{(-jk\omega_0)^2} \right]$$

$$N_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$$

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$e^{-jk\omega_0} = e^{-jk2\pi} = \cos(\omega_0 k) - j \sin(\omega_0 k) = 1$ always for any value of k

$$x(k) = \frac{-1}{jk\omega_0} - \frac{1}{(-jk\omega_0)^2} + \frac{1}{(-jk\omega_0)^2}$$

$$x(k) = \frac{-1}{jk\omega_0} = \frac{j}{k\omega_0} \quad (\because \frac{-1}{jk\omega_0} = \frac{j}{k\omega_0})$$

$$\text{If } k=0; x(k) = \int_0^t e^{-jk\omega_0 t} dt = \int_0^t e^{-j0} dt$$

$$= \left[\frac{t^2}{2} \right]_0^t = \frac{t^2}{2}$$

$$x(k) = j \frac{1}{k\omega_0}, \quad k \neq 0.$$

$$x(k) = \begin{cases} j \frac{1}{k\omega_0}; & k \neq 0 \\ y_2; & k = 0 \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{\infty} x(k) e^{jk\omega_0 t} = y_2 + \sum_{k=-\infty}^{\infty} j \frac{1}{k\omega_0} e^{jk\omega_0 t} \quad (\because \text{except } k=0)$$

To obtain Magnitude & phase spectrum

$$|x(k)| = \sqrt{(\text{real term})^2 + (\text{coefficient of imaginary term})^2}$$

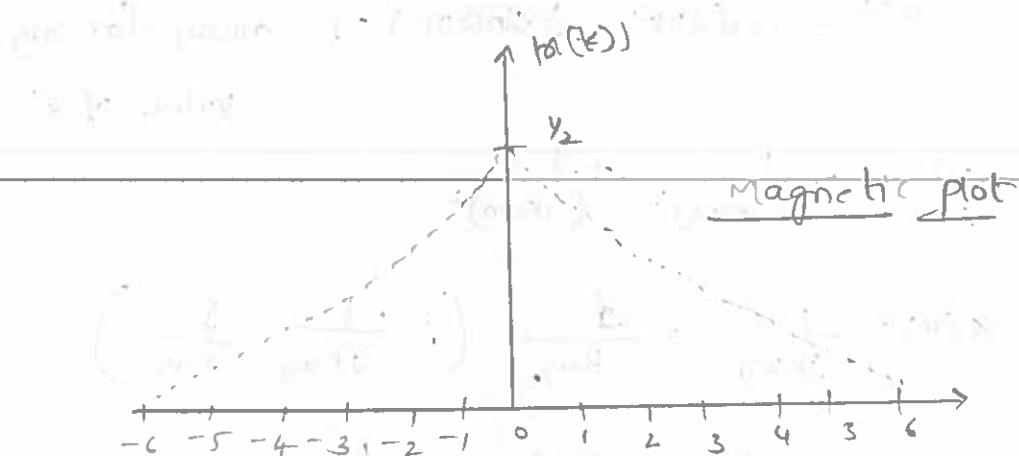
$$(\because x(k) = \text{real} + j \frac{1}{k\omega_0} = \text{real} + j \text{imaginary})$$

$$|x(k)| = \sqrt{0^2 + \frac{1}{(k\omega_0)^2}} = \sqrt{\frac{1}{(k\omega_0)^2}}$$

$$|x(k)| = \sqrt{\frac{1}{k\omega_0}} = \frac{1}{\omega_0 k}$$

$$|x(k)| = \begin{cases} \frac{1}{\omega_0 k}; & k \neq 0 \\ y_2; & k = 0 \end{cases}$$

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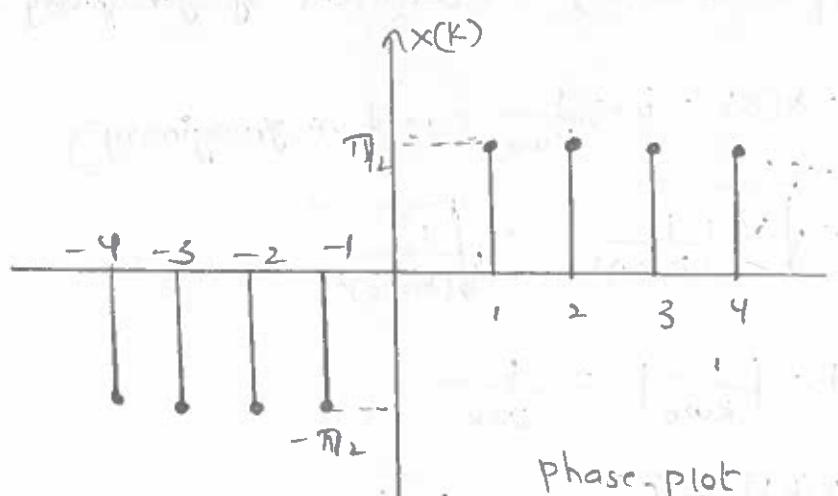
Phase spectrum:

$$|x(k)| = -\tan^{-1}\left(\frac{\text{Imaginary}}{\text{real}}\right)$$

$$= -\tan^{-1}\left(\frac{1}{\frac{k\omega_0}{\omega}}\right)$$

$$= -\pi/2$$

$$\underline{|x(k)|} = \begin{cases} \pi/2 & ; k > 0 \\ -\pi/2 & ; k < 0 \\ 0 & ; k = 0 \end{cases}$$



Fourier transform

- Non-periodic signals (A periodic) can be represented with the help of Fourier transform.
- It provides effective reversible link between frequency domain & time domain representation of the signal.
- For non-periodic signals $T_0 \rightarrow \infty$, $\omega_0 \approx 0$
 \therefore Spacing between the spectral components becomes infinite & hence spectrum appears to be continuous.
- The Fourier transform can be developed by finding Fourier series of periodic function & then tending $T \rightarrow \infty$.
- If $T \rightarrow \infty$ then $f(t)$ becomes non periodic signal.

Deriving Fourier transform from Fourier series:

Exponential form of Fourier series is given by

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{jn\omega_0 t} \quad (\text{Exponential Fourier series of synthesis})$$

$$f_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j n \omega_0 t} dt \quad (\text{Analysis Series of Exponential Fourier series})$$

For non periodic signal,

$$f_n = \int_{-\infty}^{\infty} f(t) e^{-j n \omega_0 t} dt \rightarrow 0$$

$$\text{Here } \omega = \sqrt{\frac{2\pi}{CT_s}}$$

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt} \rightarrow ②$$

Fourier transform.

Comparing eq² ① & ②

$$Tf_n = F(\omega)$$

$$f_n = \frac{1}{T} F(\omega)$$

Substituting above condⁿ in ④

$$Tf(t) = \sum_{n=-\infty}^{\infty} f(n\omega_0) e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} (f_n \omega_0) \right] e^{jn\omega_0 t}$$

$$f(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} f(n\omega_0) e^{jn\omega_0 t}$$

$$\text{But } \omega_0 = \frac{2\pi}{T} \text{ (as } T = \frac{2\pi}{\omega_0})$$

$$f(t) = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} f(n\omega_0) e^{jn\omega_0 t}$$

as $n \rightarrow \infty$; $n\omega_0 \rightarrow \omega$ (Approaches continuous frequency variable ω)

$$\therefore f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

\rightarrow IFT (Inverse Fourier transform)

Note: For periodic signals discrete wave form is formed and for aperiodic (non periodic) signal continuous wave form is formed.

→ Fourier transform of a signal $f(t)$ is defined as,

$$\rightarrow F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Inverse Fourier transform.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$\therefore F(\omega) = |F(\omega)| e^{j\phi(\omega)} \xrightarrow{\downarrow} \begin{matrix} \text{phase} \\ \text{Magnitude} \end{matrix}$$

By the application of Fourier transform a signal in time domain is converted into frequency domain

OR

$$x(t) = \sum_{n=-\infty}^{\infty} x(n) e^{jn\omega_0 t} \xrightarrow{\textcircled{1}} \text{(Fourier Series Analysis)}$$

$$x(n) = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t} dt \quad \text{(Fourier series synthesis)}$$

$$T x(n) = \int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t} dt \xrightarrow{\textcircled{2}}$$

$$x(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} T x(n) e^{jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega_0}$$

$$\therefore x(t) = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} T x(n) e^{jn\omega_0 t}$$

$$x(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} T x(n) e^{jn\omega_0 t} \omega_0$$

as $T \rightarrow \infty$, ω_0 becomes very small

Hence $\omega_0 \rightarrow d\omega$ & $T x(n)$ becomes continuous

$$\therefore x(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

Fourier transform of any arbitrary & standard signals

1. Fourier transform of single sided Exponential

Signal:

The single sided exponential signal is defined as,

$$x(t) = Ae^{-at}; t \geq 0; a - \text{constant}$$

By the definition of fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(j\omega) = \int_0^{\infty} Ae^{-at} e^{-j\omega t} dt$$

$$x(j\omega) = A \int_0^{\infty} e^{-(a+j\omega)t} dt \quad (\because x(\omega) = X(j\omega))$$

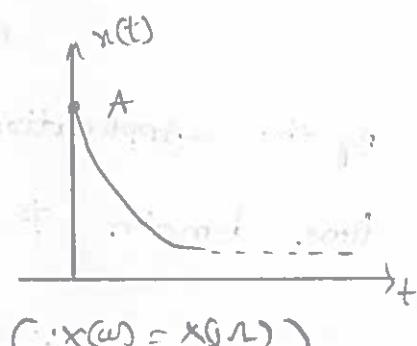
$$x(j\omega) = \left[\frac{-e^{-(a+j\omega)t}}{(a+j\omega)} \right]_0^{\infty}$$

$$x(j\omega) = \frac{Ae^{-\infty}}{+(-a+j\omega)} + \frac{Ae^0}{a+j\omega} \quad (\because e^{-\infty} = 0)$$

$$x(j\omega) = \frac{A}{(a+j\omega)}$$

$$\text{FT } (Ae^{-at}) = \frac{A}{(a+j\omega)} x(t)$$

$$(x(j\omega)) = \frac{A}{\sqrt{a^2 + \omega^2}}$$



2) Fourier transform of any arbitrary signals & standard signals:

The double sided exponential signal is defined as,

$$x(t) = Ae^{-|at|} ; \text{ for all } t$$

$$n(t) = Ae^{at} ; \text{ for } t = -\infty \text{ to } a$$

$$x(t) = -Ae^{-at} ; \text{ for } t = 0 \text{ to } \infty$$

The Fourier transform is defined as

$$X(j\omega) = \int_{-\infty}^{\infty} n(t) e^{-j\omega t} dt$$

$$X(j\omega) = \int_{-\infty}^0 A e^{at} e^{-j\omega t} dt + \int_0^{\infty} -A e^{-at} e^{-j\omega t} dt$$

$$= A \int_{-\infty}^0 e^{(a-j\omega)t} dt + \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= A \left[\frac{e^{(a-j\omega)t}}{(a-j\omega)} \right]_0^{\infty} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty}$$

$$= -A \left[\frac{e^{(a-j\omega)\infty}}{(a-j\omega)} - \frac{e^{(a-j\omega)0}}{(a-j\omega)} \right] + \frac{e^{-(a+j\omega)\infty}}{-(a+j\omega)} - \frac{e^{-(a+j\omega)0}}{-(a+j\omega)}$$

$$= A \left[\frac{1}{a-j\omega} - \frac{0}{a-j\omega} + \frac{0}{-(a+j\omega)} - \frac{1}{-(a+j\omega)} \right]$$

$$= A \left[\frac{1}{(a-j\omega)} + \frac{1}{(a+j\omega)} \right]$$

$$= \frac{(aj + 1) + (a - j\omega)}{(a - j\omega)(a + j\omega)}$$

$$= \frac{(a + j\omega) + (a - j\omega)}{(a^2 + (\omega)^2)} \quad (j^2 = -1)$$

$$= \frac{(a + j\omega) + (a - j\omega)}{(a^2 - \omega^2)}$$

$$= \frac{a + j\omega + a - j\omega}{a^2 + \omega^2}$$

$$x(j\omega) = \frac{2aA}{a^2 + \omega^2}$$

Fourier transform of a constant.

let $x(t) = A$ (constant)

$$\text{FT; } n(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt < \infty$$

Let $n_i(t) =$ double sided exponential signal

$$n_i(t) = A e^{-|at|}$$

$$n_i(t) = A \lim_{a \rightarrow 0} n_i(t) \rightarrow 0$$

Taking FT on both sides of 0

$$\mathcal{F}\{n_i(t)\} = \mathcal{F}\left\{A \lim_{a \rightarrow 0} n_i(t)\right\}$$

$$x(j\omega) = \lim_{\alpha \rightarrow 0} \text{Re} \{ x(j\omega) \}$$

$$x(j\omega) = \lim_{\alpha \rightarrow 0} \text{Re} [x(j\omega)]$$

$$= \lim_{\alpha \rightarrow 0} \frac{\alpha \omega A}{\omega^2 + \alpha^2} = 0; \quad \text{for all values of } \omega \text{ except } \omega = 0$$

At $\omega = 0$; the above eqn represents an impulse function with magnitude 'k'.

$$\therefore x(j\omega) = k \delta(\omega); \omega \neq 0$$

$$= 0; \omega \neq 0$$

The magnitude 'k' can be written as

$$k = \int_{-\infty}^{\infty} \frac{2\omega A}{\omega^2 + \alpha^2} d\omega$$

$$= 2\omega A = \int_{-\infty}^{\infty} \frac{2}{\alpha^2 + \omega^2} d\omega \rightarrow \textcircled{1}$$

$$\int \frac{1}{\omega^2 + \alpha^2} d\omega = \frac{1}{\alpha} \tan^{-1}(\omega/\alpha) \rightarrow \textcircled{2}$$

Compare \textcircled{1} & \textcircled{2} eqns.

$$k = 2\omega A \cdot \left[\frac{1}{\alpha} \tan^{-1}\left(\frac{\omega}{\alpha}\right) \right]_{-\infty}^{\infty}$$

here $\omega = \alpha$, $\omega = -\alpha$

$$k = \frac{2\omega A}{\alpha} \left[\tan^{-1}\left(\frac{\omega}{\alpha}\right) - \tan^{-1}\left(-\frac{\omega}{\alpha}\right) \right]$$

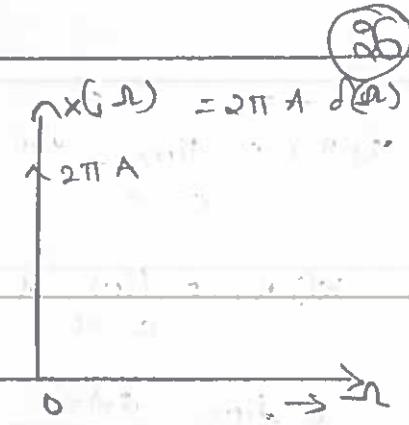
$$k = 2A \left[\tan^{-1}\infty - \tan^{-1}(-\infty) \right]$$

$$k = 2\pi [\pi I_2 + \pi I_2]$$

$$\approx 2\pi \left(\frac{2\pi}{2}\right)$$

$$\approx 2\pi A$$

$$\therefore f(\omega) = 2\pi A \delta(\omega)$$



Fourier transform of unit step function:

Let

$$u(t) = u(t) = \begin{cases} 1; t \geq 0 \\ 0; t < 0 \end{cases}$$

$$\text{sgn}(t) = u(t) - 1$$

$$\text{sgn}(t) + 1 = u(t)$$

$$x(t) = u(t) = \frac{1}{2} [\text{sgn}(t) + 1]$$

Taking FT on both sides.

$$x(\omega) = \text{FT} \left\{ \frac{1}{2} [\text{sgn}(t) + 1] \right\}$$

$$= \frac{1}{2} \text{FT}\{1\} + \frac{1}{2} \text{FT}\{\text{sgn}(t)\}$$

$$= \frac{1}{2} 2\pi \delta(\omega) + \frac{1}{2} \left(\frac{2}{j\omega} \right) \quad \left[: \text{FT}\{\text{sgn}(t)\} = \frac{2}{j\omega} \right]$$

$$\boxed{\text{FT}[u(t)] = \frac{\pi \delta(\omega)}{j\omega}}$$

Fourier transform of sinusoidal signal:

$$\text{Let } x(t) = A \sin \omega_0 t$$

$$\text{LFT, } \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$x(t) = \frac{A}{2j} [e^{j\omega_0 t} - e^{-j\omega_0 t}]$$

Taking FT on both sides

$$F\{x(t)\} = F\left\{\frac{A}{2j} [e^{j\omega_0 t} - e^{-j\omega_0 t}]\right\}$$

$$X(j\omega) = \frac{A}{2j} [F(e^{j\omega_0 t}) - F(e^{-j\omega_0 t})]$$

LFT,

$$\frac{1}{2} e^{j\omega_0 t} \xleftrightarrow{FT} \pi \delta(\omega - \omega_0) \Rightarrow e^{j\omega_0 t} \xleftrightarrow{FT} 2\pi \delta(\omega - \omega_0)$$

$$X(j\omega) = \frac{A}{2j} [2\pi \delta(\omega - \omega_0) - 2\pi \delta(\omega + \omega_0)]$$

$$X(j\omega) = \frac{A}{j} [\pi \delta(\omega - \omega_0) - \pi \delta(\omega + \omega_0)]$$

$$F(A \sin \omega_0 t) = \frac{A\pi}{j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

Fourier transform of cosine signal

$$\text{Let } x(t) = A \cos \omega_0 t$$

WFT, $\cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

$$x(t) = A \frac{1}{2} [e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

Taking FT on both sides

$$FT \{x(t)\} > FT \{A \frac{1}{2} [e^{j\omega_0 t} + e^{-j\omega_0 t}]\}$$

$$X(j\omega) = A \frac{1}{2} [F(e^{j\omega_0 t}) + F(e^{-j\omega_0 t})]$$

WFT,

$$\frac{1}{2} e^{j\omega_0 t} \xleftrightarrow{\text{FT}} \pi \delta(\omega - \omega_0)$$

$$e^{j\omega_0 t} \xleftrightarrow{\text{FT}} 2\pi \delta(\omega - \omega_0)$$

$$X(j\omega) = A \frac{1}{2} [\pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)]$$

$$x(j\omega) = A [\pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)]$$

$$F(A \cos \omega_0 t) = A\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

Properties of Fourier transform

1) Linearity:

Statement: If $x(t) \xleftrightarrow{\text{FT}} X(\omega)$ & $y(t) \xleftrightarrow{\text{FT}} Y(\omega)$

then, $z(t) = a x(t) + b y(t) \xleftrightarrow{\text{FT}} Z(\omega) = a X(\omega) + b Y(\omega)$.

i.e,

The Fourier transform of linear combination of the signals is equal to linear combination of their Fourier transform.

FT is given by,

$$Z(\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} [a x(t) + b y(t)] e^{-j\omega t} dt$$

$$Z(\omega) = a \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$\boxed{Z(\omega) = a X(\omega) + b Y(\omega)}$$

Time Shift:

Statement: If $y(t) \xleftrightarrow{\text{FT}} Y(\omega)$

$$y(t) = y(t-t_0) \xleftrightarrow{\text{FT}} Y(\omega) = e^{-j\omega t_0} Y(\omega)$$

$$\text{WKT, } Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} y(t-t_0) e^{-j\omega t} dt$$

~~Unit-2 : 29/50~~

$$\text{put } t-t_0 = \tau \Rightarrow t = \tau + t_0$$

$$dt = d\tau$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega(t+t_0)} dt$$

$$= \int_{-\infty}^{\infty} y(t) e^{-j\omega t} e^{-j\omega t_0} dt$$

$$Y(\omega) = e^{-j\omega t_0} \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$Y(\omega) = e^{-j\omega t_0} Y(\omega)$$

Frequency shift:

Statement: If $x(t) \xrightarrow{\text{FT}} X(\omega)$ then,

$$y(t) = e^{j\beta t} x(t) \xrightarrow{\text{FT}} Y(\omega) = X(\omega - \beta)$$

$$\text{so, } Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \beta)t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{j\beta t} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \beta)t} dt$$

$$= X(\omega - \beta)$$

i.e., it states that by shifting the frequency by β in frequency domain is equivalent to multiplying with time domain signal by $e^{j\beta t}$.

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Time Scaling:

Statement:

If $x(t) \xrightarrow{\text{FT}} X(\omega)$ then

$$y(t) = x(at) \xrightarrow{\text{FT}} Y(\omega) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

compression of a signal in time domain is equivalent to expansion in frequency domain and vice versa.

F.T is given by

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

$$\text{let } at=t \Rightarrow t=a$$

$$dt = 1/a dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j\omega t/a} \frac{1}{a} dt$$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} x(t) e^{-j(\omega/a)t} dt$$

$$= \frac{1}{|a|} X(\omega/a).$$

Unit-2: 31/50

Frequency differentiation:

If Fourier transform of

$$x(t) \xrightarrow{FT} X(\omega) \text{ then,}$$

$$-jt X(t) \leftrightarrow \frac{d}{d\omega} X(\omega).$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

diff w.r.t ω , we get both sides,

$$\frac{d}{d\omega} X(\omega) = \frac{d}{d\omega} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{d}{d\omega} X(\omega) = \int_{-\infty}^{\infty} x(t) \frac{d}{d\omega} (e^{-j\omega t}) dt$$

$$= \int_{-\infty}^{\infty} x(t) (-j e^{-j\omega t}) dt$$

$$= -jt \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{d}{d\omega} X(\omega) = -jt X(\omega)$$

Time differentiation:

$$F(x(t)) = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$F\left(\frac{d}{dt} x(t)\right) = \int_{-\infty}^{\infty} \frac{d}{dt} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} \left(\frac{d}{dt} x(t)\right) dt$$

$$\mathcal{F}\left(\frac{d}{dt} x(t)\right) = \left[e^{-j\omega t} x(t)\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (-j\omega) e^{-j\omega t} x(t) dt$$

$$\therefore [e^{-\infty} x(\infty)] - [e^{\infty} - x(-\infty)] + j\omega \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad (\because e^{\infty} = 0)$$

$$\boxed{\mathcal{F}\left\{\frac{d}{dt} x(t)\right\} = j\omega x(\omega)}$$

convolution:

If $x(t) \xrightarrow{\text{FT}} X(\omega)$ & $y(t) \xrightarrow{\text{FT}} Y(\omega)$ then,

$$z(t) = x(t) * y(t) \xrightarrow{\text{FT}} z(\omega) = X(\omega) \cdot Y(\omega).$$

$$\text{FT: } z(\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} [x(t) * y(t)] e^{-j\omega t} dt$$

$$\text{LKT, } x(t) * y(t) = \int_{-\infty}^{\infty} y(\tau) y(t-\tau) d\tau$$

$$z(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \right] e^{-j\omega t} dt$$

$$\text{put } t-\tau = \alpha \Rightarrow t = \tau + \alpha$$

$$dt = d\alpha$$

$$z(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\alpha) y(\alpha) d\alpha \right] e^{-j\omega(t+\alpha)} d\alpha$$

$$\text{put } t+\alpha = \omega \Rightarrow t = \omega - \alpha$$

$$dt = -d\alpha$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\alpha) y(\alpha) d\alpha \right] e^{-j\omega t} e^{-j\omega \alpha} d\alpha$$

(Part 2: 33/50)

$$\int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^t y(\alpha) e^{-j\omega\alpha} d\alpha \right] e^{-j\omega t} dt$$

$$x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \cdot \int_{-\infty}^{\infty} y(\alpha) e^{-j\omega\alpha} d\alpha$$

$$\therefore x(\omega) = X(\omega) \cdot Y(\omega)$$

Integration:

If $x(t) \xrightarrow{FT} X(\omega)$ then,

$$\int_{-\infty}^t x(\tau) d\tau \xrightarrow{FT} \frac{1}{j\omega} X(\omega).$$

Integration in time domain represents smoothing in time domain this smoothing in time corresponds to de-emphasising the high frequency components of the signal.

Proof: Let us consider $x(t) = \frac{d}{dt} \left[\int_{-\infty}^t x(\tau) d\tau \right]$

Taking Fourier transform on both sides.

$$FT \{ x(t) \} = FT \left\{ \frac{d}{dt} \left(\int_{-\infty}^t x(\tau) d\tau \right) \right\}$$

$$FT \{ x(t) \} = j\omega \left\{ FT \left[\int_{-\infty}^t x(\tau) d\tau \right] \right\}$$

Since, by differentiation property,

$$\frac{d}{dt} x(t) \xrightarrow{FT} j\omega X(\omega)$$

$$\frac{1}{j\omega} X(\omega) = FT \int_{-\infty}^t x(\tau) d\tau$$

$$\int_{-\infty}^t x(t) dt \xleftrightarrow{FT} \frac{1}{j\omega} x(\omega).$$

Modulation:

If $x(t) \xleftrightarrow{FT} X(\omega)$ & $y(t) \xleftrightarrow{FT} Y(\omega)$ then,

$$z(t) = x(t) y(t) \xleftrightarrow{FT} Z(\omega) = \frac{1}{2\pi} [X(\omega) * Y(\omega)]$$

i.e Modulation in time domain corresponds to convolution of spectrum in frequency domain.

IIFT {inverse fourier transform}

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$z(t) = \int_{-\infty}^{\infty} z(\omega) e^{-j\omega t} d\omega = \int_{-\infty}^{\infty} x(\omega) y(\omega) e^{-j\omega t} d\omega \rightarrow FT$$

$$z(\omega) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} x(\lambda) e^{j\lambda t} d\lambda \right] y(t) e^{-j\omega t} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\lambda) \left[\int_{-\infty}^{\infty} y(t) e^{-j(\omega-\lambda)t} dt \right] d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\lambda) Y(\omega - \lambda) d\lambda$$

$$z(\omega) = \frac{1}{2\pi} [X(\omega) * Y(\omega)]$$

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Duality:

If $x(t) \xleftrightarrow{\text{FT}} X(\omega)$.

then $x(t) \xleftrightarrow{\text{FT}} 2\pi X(-\omega)$

proof: inverse fourier transform is given by,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad [\because t \text{ is interchanged with } \omega \text{ & vice-versa}]$$

Replacing ω by $(-\omega)$ we get,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(-\omega) e^{-j\omega t} d\omega$$

$$2\pi X(-\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Q.H.S the eq? is FT of $x(t)$

i.e., $x(t) \xleftrightarrow{\text{FT}} 2\pi X(-\omega)$.

Symmetry:

let $x(t)$ be a real signal & $X(\omega) = X_R(\omega) + jX_I(\omega)$

$\rightarrow \textcircled{1}$

the $x_e(t) \xleftrightarrow{\text{FT}} X_R(\omega)$.

& $x_o(t) \xleftrightarrow{\text{FT}} jX_I(\omega)$.

Where $x_e(t)$ & $x_o(t)$ are even & odd parts of $x(t)$

proof: we have

$$x(t) \xleftrightarrow{\text{FT}} X_R(\omega) + jX_I(\omega)$$

since $x(t)$ is real,

Unit-2: 36/50

$$x(t) \xrightarrow{FT} x(\omega) = x_p(\omega) - jx_I(\omega) \rightarrow ②$$

even part is given as, $x_e(t) = V_2 [x(t) + x^*(t)]$

$$x_e(t) \xrightarrow{FT} V_2 [x(\omega) + x(-\omega)] \rightarrow ③$$

substituting Eqn ① & ② in Eqn ③

$$x_e(t) \xrightarrow{FT} V_2 [x_p(\omega) + j\omega_2(\omega) + x_p(\omega) - jx_I(\omega)]$$

$$x_e(t) \xrightarrow{FT} V_2 2x_p(\omega) \quad \because x_e(t) \xrightarrow{FT} x_p(\omega)$$

odd part is given as,

$$x_o(t) = V_2 [x(t) - x^*(t)]$$

$$x_o(t) \xrightarrow{FT} V_2 [x_p(\omega) + jx_I(\omega) - x_p(\omega) + jx_I(\omega)]$$

$$x_o(t) \xrightarrow{FT} V_2 2jx_I(\omega)$$

$$x_o(t) \xrightarrow{FT} 2jx_I(\omega)$$

Parsvals theorem or Layleigh theorem:

Statement: If $x(t) \xrightarrow{FT} X(\omega)$ then,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t) x^*(t) dt \rightarrow ④$$

Unit-2: 37/50

$$\text{IFT: } x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega$$

Taking conjugate on both sides.

$$x^*(t) = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{-j\omega t} d\omega \right]^*$$

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^*(\omega) e^{-j\omega t} d\omega \rightarrow ②$$

Substituting eq ② in eq ①,

$$E = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} x^*(\omega) e^{-j\omega t} d\omega \right] dt$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^*(\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^*(\omega) x(\omega) d\omega$$

$$\text{But } \omega = 2\pi f$$

$$d\omega = 2\pi df$$

$$\text{hence, } E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |x(f)|^2 2\pi df$$

$$E = \int_{-\infty}^{\infty} |x(f)|^2 df$$

(39)

$$x(t) = \delta(t)$$

\hat{x} is given by,

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt \rightarrow \Theta$$

using shifting property, $\int_{-\infty}^{\infty} x(t) \cdot d(t-t_0) dt = x(t_0) \rightarrow \Theta$

compare Eqn ① & Eqn ②

Hence in the above eqn $x(t) = e^{-j\omega t}$

Hence, $x(\omega) = e^{-j\omega t}$ $d(t) \xleftrightarrow{FT} 1$	$\& t_0 = 0$ $= 1$ $t = 0.$
----------------------------------------------------------------------	-----------------------------------

Q2

$$\text{At } x(t) = 1$$

$$\epsilon = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} (1)^2 dt = \infty \rightarrow \text{this mean's Dirichlet's condn is}$$

not satisfied, but its \hat{x} can be calculated using duality property.

$$\text{WKT, } \delta(t) \xleftrightarrow{FT} 1 \text{ ie, } x(\omega) = 1$$

\therefore Fourier transform of $\delta(t) \xleftrightarrow{FT} x(\omega)$

Duality property states that,

~~Unit-2: 39/50~~

AO

$x(t) \xleftrightarrow{FT} 2\pi \delta(\omega)$; but $x(t)=1$ &
 $\delta(\omega) = \delta(\omega)$.

$\therefore 1 \not\xleftrightarrow{FT} 2\pi \delta(\omega)$.

WKT, $\delta(\omega)$ will be even function of 'b'.
 Since, it is an impulse function.

$\therefore \delta(-\omega) = \delta(\omega)$.

$$\boxed{\therefore 1 \xleftrightarrow{FT} 2\pi \delta(\omega)} \quad (\because x(t) \xleftrightarrow{FT} x(\omega))$$

If $x(t)=1$, then $x(\omega)=1$; then $x(\omega)=2\pi \delta(\omega)$.

Find Fourier transform of $x(t) = e^{-at} u(t)$.

$$\text{WKT, } x(t) = \begin{cases} 1; & t \geq 0 \\ 0; & t < 0 \end{cases}$$

FT is given by,

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$X(\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

$$X(\omega) = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-a-j\omega} \right]_0^{\infty}$$

Unitary Holso

$$= \frac{-1}{(a+j\omega)} (e^{-\infty} - e^0)$$

$$= \frac{-1}{a+j\omega} (-1)$$

$$X(\omega) = \frac{1}{a+j\omega}$$

$$\therefore e^{-at} u(t) \xleftrightarrow{\text{FT}} \frac{1}{a+j\omega}$$

To obtain magnitude plot:

$$X(\omega) = \frac{1}{a+j\omega} \times \frac{a-j\omega}{a-j\omega}$$

$$X(\omega) = \frac{a-j\omega}{a^2 + (\omega)^2} \quad (\text{Here } j^2 = -1)$$

$$X(\omega)^2 = \frac{a-j\omega}{a^2 + (\omega)^2}$$

$$X(\omega) = \frac{a}{a^2 + (\omega)^2} \rightarrow \frac{\omega}{a^2 + \omega^2}$$

Magnitude

$$|X(\omega)| = \sqrt{\left(\frac{a}{a^2 + \omega^2}\right)^2 + \left(\frac{\omega}{a^2 + \omega^2}\right)^2}$$

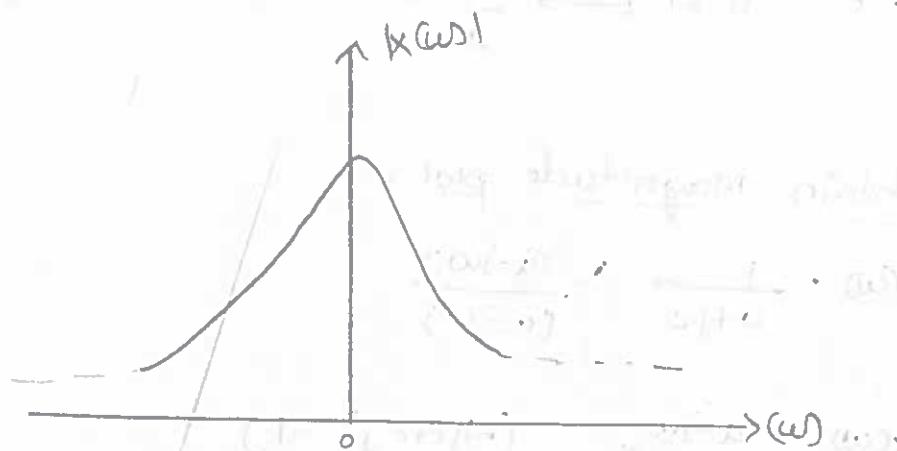
$$|X(\omega)| = \sqrt{\frac{a^2 + \omega^2}{(a^2 + \omega^2)^2}}$$

(A²)

$$|X(\omega)| = \sqrt{\frac{1}{a^2 + \omega^2}}$$

where $a=1 = \sqrt{\frac{1}{1+\omega^2}}$

if $a=1$, *



To obtain phase shift: plot:

(Phase angle)

$$|X(\omega)| = \tan^{-1} \left(\frac{\text{Imaginary part}}{\text{real part}} \right)$$

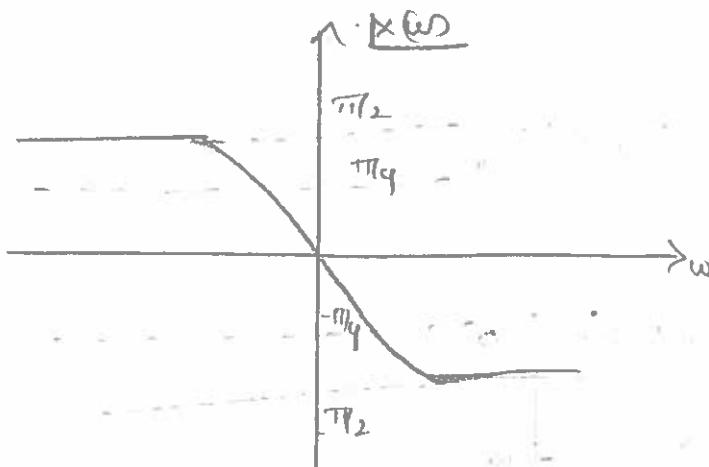
$$= -\tan^{-1} \left(\frac{\omega}{\frac{a^2 + \omega^2}{a}} \right)$$

$$|X(\omega)| = \tan^{-1} \left(\frac{\omega}{a} \right)$$

∴

unit-2: also

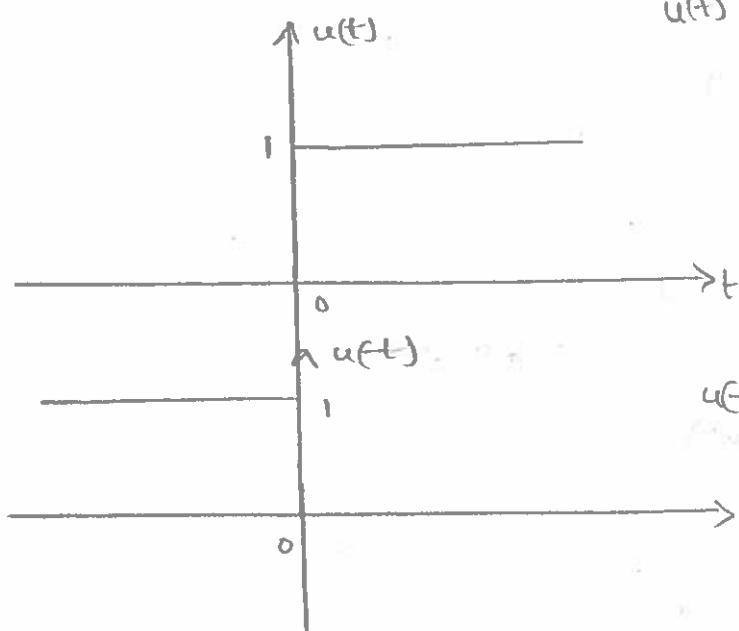
If $a=1$, then



Find Fourier transform of $x(t) = e^{-at} u(-t)$

$$u(t) = \begin{cases} 1; & t \leq 0 \\ 0; & t > 0 \end{cases}$$

$$u(t) = \begin{cases} 1; & t \geq 0 \\ 0; & t < 0 \end{cases}$$



$$u(t) = \begin{cases} 1; & t \leq 0 \\ 0; & t > 0 \end{cases}$$

$X(\omega)$ is given by,

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(-t) e^{-j\omega t} dt$$

Unit-2 : 43/50

$$x(\omega) = \int_{-\infty}^0 e^{-at} (1) e^{-j\omega t} dt$$

$$= \int_{+\infty}^0 e^{-(a+j\omega)t} dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_{0, \infty}$$

$$= \frac{e^0}{-(a+j\omega)} - \frac{e^{-\infty}}{-(a+j\omega)}$$

$$= \frac{-e^{-\infty}}{-(a+j\omega)}$$

$$= \frac{-1}{+ (a+j\omega)}$$

$$= \frac{1}{-(a+j\omega)} \quad (e^0 - e^{-\infty})$$

$$= \frac{1}{-(a+j\omega)}$$

Magnitude plot will be same but phase plot will be reflected version.

$$n(t) e^{-at}$$

$$\text{ie } x(t) = \begin{cases} e^{-at}; & t > 0 \\ 1 & ; t=0 \\ e^{at} & ; t < 0 \end{cases}$$

$$x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_0^{\infty} e^{-at} e^{-j\omega t} dt + \int_0^{\infty} e^{-j\omega t} dt$$

$$+ \int_{-\infty}^0 e^{at} e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t} dt + \int_0^{\infty} e^{-j\omega t} dt + \int_{-\infty}^0 e^{(a-j\omega)t} dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty} + \left[\frac{e^{-j\omega t}}{-j\omega} \right]_0^{\infty} + \left[\frac{e^{(a-j\omega)t}}{(a-j\omega)} \right]_{-\infty}^0$$

$$= \frac{e^{-(a+j\omega)\infty}}{-(a+j\omega)} - \frac{e^0}{-(a+j\omega)} + \frac{e^{-j\omega\infty}}{-j\omega} - e^0 + \frac{1}{a-j\omega}$$

$$\left[\frac{e^{(a-j\omega)\infty}}{(a-j\omega)} \right]$$

$$= e^{\infty} - \frac{e^0}{-(a+j\omega)} + \frac{1-1}{-j\omega} + \frac{e^0}{a-j\omega} - \frac{e^{-\infty}}{a-j\omega}$$

$$= \frac{1}{a+j\omega} + \frac{0}{-j\omega} + \frac{1}{a-j\omega}$$

$$= \frac{a-j\omega + a+j\omega}{a^2+\omega^2}$$

$$= \frac{2a}{a^2+\omega^2}$$

~~WTF :- 2 : 45/50~~

Magnitude:

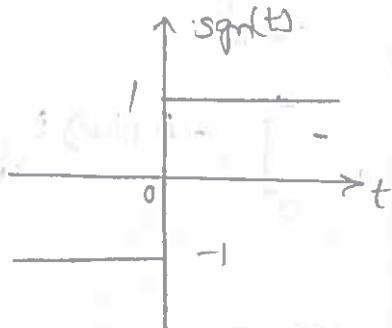
$$x(\omega) = \frac{2a}{a^2 + \omega^2} \times \frac{a^2 - \omega^2}{a^2 - \omega^2}$$

$$= \frac{2a(a^2 - \omega^2)}{(a^2 + \omega^2)^2}$$

Fourier transform of unit impulse & signum function

Signum Function:

$$\text{Sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \\ 0 & t = 0 \end{cases}$$



$$\text{Sgn}(t) = \lim_{a \rightarrow 0} [e^{-at} u(t) - e^{at} u(-t)]$$

Let

$$x(t) = \text{Sgn}(t) = \lim_{a \rightarrow 0} [e^{-at} u(t) - e^{at} u(-t)]$$

$$x(\omega) = x(j\omega) = \int_{-\infty}^{\infty} i(xt) e^{-j\omega t} dt$$

$$x(j\omega) = \int_{-\infty}^{\infty} \lim_{a \rightarrow 0} [e^{-at} u(t) - e^{at} u(-t)] e^{-j\omega t} dt$$

$t \geq 0$

$$x(j\omega) = \lim_{a \rightarrow 0} \left[\int_0^{\infty} e^{-(a+j\omega)t} dt - \int_{-\infty}^0 e^{(a-j\omega)t} dt \right]$$

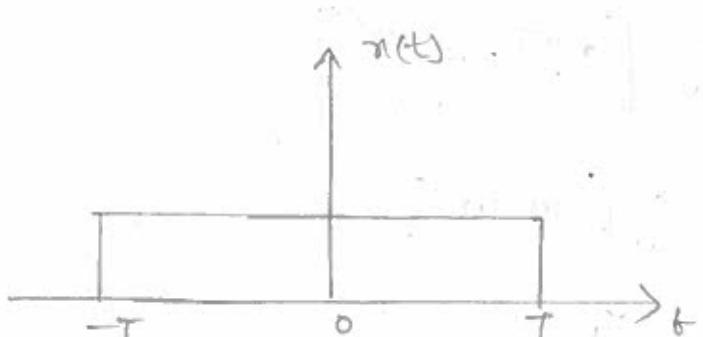
unit-2: 46/50

$$\begin{aligned}
 x(j\omega) &= \lim_{\alpha \rightarrow 0} \left[\left(\frac{e^{-(a+j\omega)t}}{-a+j\omega} \right)_0^\infty - \left(\frac{e^{(a-j\omega)t}}{(a-j\omega)} \right)_{-\infty}^0 \right] \\
 &= \lim_{\alpha \rightarrow 0} \left[\frac{1}{a+j\omega} (e^{-a+j\omega t})_0^\infty - \frac{1}{a-j\omega} (e^{a-j\omega t})_{-\infty}^0 \right] \\
 &= \lim_{\alpha \rightarrow 0} \left[\frac{1}{a+j\omega} (0-1) - \frac{1}{a-j\omega} (1-0) \right] \\
 &= \lim_{\alpha \rightarrow 0} \left(\frac{1}{a+j\omega} - \frac{1}{a-j\omega} \right) \\
 x(j\omega) &= \frac{1}{j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega}
 \end{aligned}$$

$$\text{sgn}(t) \xrightarrow{\text{FT}} \frac{2}{j\omega}$$

Fourier rectangular series:

prove (oo) show that the given signal is equal to $\frac{2T}{\pi} \text{sinc}(\omega T)$.



$$x(\omega) = \int_{-\infty}^{\infty} n(t) e^{-j\omega t} dt$$

~~WAT 2015~~

$$n(t) = \begin{cases} 1; & -T \leq t \leq T \\ 0; & \text{otherwise} \end{cases}$$

$$x(\omega) = \int_{-T}^T 1 \cdot e^{-j\omega t} dt$$

$$x(\omega) = \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-T}^T$$

$$\begin{aligned} x(\omega) &= \frac{-1}{j\omega} [e^{j\omega T} - e^{-j\omega T}] \\ &= \frac{1}{j\omega} (e^{j\omega T} - e^{-j\omega T}) \end{aligned}$$

$$\text{Let } \theta = \frac{e^{j\omega T} - e^{-j\omega T}}{2j}$$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

divide & multiply by 2

$$x(\omega) = \frac{1}{\omega} \frac{2}{2j} [e^{j\omega T} - e^{-j\omega T}]$$

$$x(\omega) = \frac{2}{\omega} \left[\frac{e^{j\omega T} - e^{-j\omega T}}{2j} \right]$$

$$x(\omega) = \frac{2}{\omega} [\sin(\omega T)]$$

$$\text{Let } \theta = \frac{\sin(\pi \theta)}{\pi \theta} \quad (\frac{\sin \theta}{\theta} \approx \sin \theta)$$

$$x(\omega) = \frac{2}{\omega} \sin(\omega T) \frac{T}{T}$$

$$= 2T \frac{\sin \omega T}{\omega T}$$

$$\therefore x(\omega) = 2T \operatorname{sinc}(\omega T)$$

$$1) \pi(t) = \sin \omega_0 t$$

$$\begin{aligned} \sin \omega_0 t &= \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \\ &= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} \end{aligned}$$

DFT, $1 \xleftrightarrow{\text{FT}} 2\pi \delta(\omega)$

$\frac{1}{2} \xleftrightarrow{\text{FT}} \pi \delta_j(\omega)$.

frequency shifting property states.

$$\frac{1}{2j} e^{j\omega_0 t} \xleftrightarrow{\text{FT}} \frac{\pi}{j} \delta(\omega - \omega_0).$$

$$\frac{1}{2j} e^{-j\omega_0 t} \xleftrightarrow{\text{FT}} \frac{\pi}{j} \delta(\omega + \omega_0)$$

$$\text{FT} \cdot (\pi(t)) = \text{FT} \left\{ \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} \right\}$$

$$= \pi \delta(\omega_0 - \omega) - \pi \delta(\omega_0 + \omega)$$

$$= \pi \left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0) \right]$$

$$2) \pi(t) = \cos \omega_0 t$$

$$\cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$= \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}.$$

DFT, $1 \xleftrightarrow{\text{FT}} 2\pi \delta(\omega)$

$\frac{1}{2} \xleftrightarrow{\text{FT}} \pi \delta(\omega)$

frequency shifting property

~~Unit-2 % 49/50~~

$$\frac{1}{2} e^{j\omega_0 t} \xleftrightarrow{\text{FT}} \pi \delta(\omega - \omega_0)$$

$$\frac{1}{2} e^{-j\omega_0 t} \xleftrightarrow{\text{FT}} \pi \delta(\omega + \omega_0)$$

$$\text{FT} \{ x(t) \} = \text{FT} \left\{ \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} \right\}$$

$$= \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$$

$$= \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

